CIS 2910 Lab 6 – Solutions to the Counting Problems

The All-Spelled-Out Edition!

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1 Welcome back to combinatorics!

A few students asked for solutions, so here they are!

1. Books on a bookshelf

   (a) There are six books standing upright on a bookshelf. How many ways are there to arrange them on this bookshelf?

   The solution is $6! = 720$**.** To reason about this, it might help to think of the six spots of the bookshelf with a diagram like the one below ...

$$= \_\ \ \_\ \ \_\ \ \_\ \ \_\ \ \_ \tag{1}$$

$$= (6) \cdot (5) \cdot (4) \cdot (3) \cdot (2) \cdot (1) \tag{2}$$

   In the first spot, we can put any of the six books – but in the second spot, we have used one book so only five books are available to select. In the third position, there are four books to select. In the fourth, there are three books. In the fifth, two books and in the last, there is only one book left. You can recognize this relationship as a factorial of the number of books we started with: six; thus our answer is six factorial = 720.

   (b) There is exactly one red book among your six books. How many arrangements of the six books are there such that the red book is furthest to the left?

   The solution is $5! = 120$**.** We can start with the same diagram as in the previous question with the stipulation that the first spot is always taken by the red book (thus there is always only one book to select in that spot).

$$= (1)\ \ \_\ \ \_\ \ \_\ \ \_\ \ \_ \tag{3}$$

$$= (1) \cdot (5) \cdot (4) \cdot (3) \cdot (2) \cdot (1) \tag{4}$$

In the second position, you select a book from the five remaining books. In the third, select a book from four and onward until the last position where only one book remains. We recognize this as five factorial.

(c) There are exactly two blue books among your six books. How many arrangments of the six books are there such that the two blue books are standing together?

The solution is $2(5!) = 240$. We reason about this by breaking this into two subproblems. First, we treat the two blue books as a single unit, and ask how many ways there are to arrange that single unit amongst the other books. Second, we ask how many ways there are to arrange the two blue books amongst themselves.

i. *Arranging the two blue books amongst the other books*

The solution to this subproblem is $5!$.

$$= [\_ \quad \_] \quad \_ \quad \_ \quad \_ \quad \_ \tag{5}$$

$$= [\_] \quad \_ \quad \_ \quad \_ \quad \_ \tag{6}$$

$$= (5) \cdot (4) \cdot (3) \cdot (2) \cdot (1) \tag{7}$$

If we treat the two blue books as a unit, then there are a total of five objects to arrange, as there are four remaining books. Arranging five objects is the same as five factorial $(5!) = 120$.

ii. *Arranging the two books amongst themselves*

The solution to this subproblem is 2.

$$= [\_ \quad \_] \quad \_ \quad \_ \quad \_ \quad \_ \tag{8}$$

$$= [\_ \quad \_] \tag{9}$$

$$= (2) \cdot (1) \tag{10}$$

Arranging two objects is the same as $(2!) = 2$.

Finally, we multiply the results of the two subproblems together to get $2(5!)$.

2. Numerals and letters

(a) How many ten digit binary values have the character `1` in them?

The solution is $2^{10} - 1 = 1023$. The easiest way to reason about this problem is to rephrase it as: *the total number of ten-digit binary values minus the total number of binary values without* *the numeral* 1. There are 2^{10} ten-digit binary values. There is only one ten-digit binary value made with all zeros: 0000000000.

(b) How many five digit hexadecimal values have the character `F` in them?

The solution is $16^5 - 15^5$**.** We can reason about this in the same way as the last problem, by rephrasing it as: *the total number of five-digit hexadecimal values minus the total number of five-digit hexadecimal values without the numeral* F. There are 16^5 five-digit hexadecimal values; recall that the base sixteen is because there are sixteen characters in the alphabet $[0-9A-F]$. There are 15^5 five-digit hexadecimal values without the numeral F. The base is because there are fifteen characters in such an alphabet $[0-9A-E]$.

(c) How many six digit hexadecimal values have the substring `BEEF` in them?

The solution is $3(16^2) = 768$**.** We should break this problem apart into three pieces. First, the number of ways we can arrange the substring `BEEF` amongst six characters; second, the number of ways we can make the substring `BEEF`; and third, the number of ways we can select the remaining characters.

i. *Arranging the substring* `BEEF` *amongst six characters.*

The solution to this subproblem is 3**.**

$$= [B \quad E \quad E \quad F] \quad \_ \quad \_ \qquad \text{– arrangement 1} \tag{11}$$

$$= \_ \quad [B \quad E \quad E \quad F] \quad \_ \qquad \text{– arrangement 2} \tag{12}$$

$$= \quad \_ \quad \_[B \quad E \quad E \quad F] \qquad \text{– arrangement 3} \tag{13}$$

$$\tag{14}$$

ii. *Making the substring* `BEEF`*.*

The solution to this subproblem is 1**.**
There is only one way to spell `BEEF`. The two letter (`E`)s do not affect our count since rearranging them is not tied to rearranging actual objects. We can say that we are drawing letters out of the set of hexadecimal characters *with replacement*, rather than drawing them out exhaustively.

iii. *Selecting the remaining characters.*

The solution to this subproblem is $16^2 = 256$**.**
To select two hexadecimal characters, we select from $[0-9A-F]$ for one character then again from $[0-9A-F]$ for the second character. Notice that the number of characters to select from does not diminish as they are selected *with replacement*.

We multiply the results of our subproblems together to arrive at $3(16^2)$.