CIS 2910 Lab 6 – Homogeneous Recurrences and Counting

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Hello CIS 2910, and welcome to Lab 6. Today, we'll cover these topics ...

- Solving homogeneous linear recurrences
  - Synthetic multiplication (for your information)
  - Synthetic division (for your information)
- A well-known logarithmic recurrence (not tested)
- Combinatorics: Introduction or review to counting
- Lab Test 2 next week

You will have discussed these items in class before, so this lab is intended to provide you with a breakdown of each procedure followed by some practical experience. The objective is to be able to perform the skills discussed today when you encounter these kinds of problems in real life.

1 Solving homogeneous linear recurrences

Homogeneous linear recurrences are sequences such that each term is the sum of one or more preceeding terms. These preceeding terms may be scaled. When a $\theta(1)$ equation is found, we say that the recurrence is solved. For more examples, a derivation for second-order linear recurrences, and the general form for these recurrences, please see §8.2 of Rosen 7th [1].
Let us work through a binary recurrence together.

1. **Example** Let us consider the binary relation,

$$t_n = 5t_{n-1} - 6t_{n-2} \tag{1}$$
$$t_0 = 1 \tag{2}$$
$$t_1 = 6 \tag{3}$$

We can rephrase the above into our characteristic equation which shows us where to put our coefficients into a form from which we factor out our characteristic roots.

$$a_2t_2 + a_1t_1 + a_0t_0 = 0 \tag{4}$$
$$x^2 - a_1x - a_0 = 0 \qquad \text{(signs are changed.)} \tag{5}$$
$$x^2 - 5x + 6 = 0 \tag{6}$$
$$(x-2)(x-3) = 0 \tag{7}$$

The roots are $2, 3$. We now need to find the values for the constants (A, B) in our general solution.

$$t_n = Ar_1^n + Br_2^n \tag{8}$$

We know our initial two terms in the recurrence, so we substitute and solve for the values (A, B) as a system of two equations with two unknowns.

$$
\begin{align}
t_0 &= Ar_1^0 + Br_2^0 && \text{eq 1} \tag{9}\\
t_1 &= Ar_1^1 + Br_2^1 && \text{eq 2} \tag{10}\\
1 &= A + B && \text{eq 1 subtituted for term at index zero} \tag{11}\\
6 &= A(2) + B(3) && \text{eq 2 substituted for term at index one} \tag{12}\\
B &= 1 - A && \text{eq 1} \tag{13}\\
6 &= 2A + 3(1 - A) && \text{eq 2 with B substituted} \tag{14}\\
6 &= 2A + 3 - 3A \tag{15}\\
A &= -3 \tag{16}\\
B &= 1 + 3 && \text{eq 1 with A substituted} \tag{17}\\
B &= 4 \tag{18}
\end{align}
$$

Going back to our general solution, we can substitute these values and come up with our result.

$$
\begin{align}
t_n &= Ar_1^n + Br_2^n \tag{19}\\
t_n &= -3(2)^n + 4(3)^n \tag{20}
\end{align}
$$

Thus concludes the example. Work through the following three problems.

2. **Solve this Binary Recurrence**

   The following example is from [1] p. 516.

$$
\begin{align}
t_n &= t_{n-1} + 2t_{n-2} \tag{21}\\
t_0 &= 2 \tag{22}\\
t_1 &= 7 \tag{23}
\end{align}
$$

3. **Solve this Binary Recurrence**

$$t_n = 6t_{n-1} - 8t_{n-2} \tag{24}$$
$$t_0 = 2 \tag{25}$$
$$t_1 = 18 \tag{26}$$

4. **Solve this Ternary Recurrence**

$$t_n = -5t_{n-1} - 2t_{n-2} + 8t_{n-3} \tag{27}$$
$$t_0 = 6 \tag{28}$$
$$t_1 = -8 \tag{29}$$
$$t_2 = 30 \tag{30}$$

Hint: Synthetic division is useful here – try dividing in $(x - 1)$

1.1 Synthetic multiplication (shorthand for your information)

We use synthetic multiplication as a shorthand for expanding out polynomials in the form $y = (x+a_1)\cdot(x+a_2)\cdot ... \cdot (x+a_n)$. It is easier to understand with an example, than it is to understand given a description. Consider this polynomial.

$$y = (x+1)(2x+3)(x-4) \tag{31}$$

We perform synthetic multiplication as follows (comments explain the action present on each line).

$$
\begin{array}{rrrr|rl}
 & & 1 & 1 & \texttt{from} & (1x+1) \\
 & & 2 & 3 & \texttt{from} & (2x+3) \\
\hline
 & & 3 & 3 & \texttt{product of} & 3(x+1) \\
 & 2 & 2 & & \texttt{product of} & 2x(x+1) \\
\hline
 & 2 & 5 & 3 & \texttt{sum is} & 2x^2+5x+3 \\
 & & 1 & -4 & \texttt{from} & (1x+(-4)) \\
\hline
 & -8 & -20 & -12 & \texttt{product of} & -4(2x^2+5x+3) \\
2 & 5 & 3 & & \texttt{product of} & x(2x^2+5x+3) \\
\hline
2 & -3 & -17 & -12 & \texttt{sum is} & 2x^3-3x^2-17x-12
\end{array}
\tag{32}
$$

Our example expands to $2x^3 - 3x^2 - 17x - 12$.

1.2 Synthetic division (shorthand for your information)

We use synthetic division as a means to test a real root of an expanded polynomial. Just as in the division of integers, we want to guess factors that divide evenly and produce no remainders. Just as with synthetic multiplication, this is easier to understand with an example.

$$y = x^3 + 5x^2 + 2x - 8 \tag{33}$$

Let us say we think that $(+1)$ is a real root for this equation. This means that the equation has $(x - 1)$ as a factor. To test if this is true, we perform synthetic division (*left*). To make things clear, full polynomial division is also shown (*right*).

$$
\begin{array}{rrrrr}
 & & 1 & 6 & 8 \\
\hline
x-1) & 1 & 5 & 2 & -8 \\
 & 1 & -1 & & \\
 & & 6 & 2 & \\
 & & 6 & -6 & \\
 & & & 8 & -8 \\
 & & & 8 & -8 \\
 & & & & 0
\end{array}
\qquad
\begin{array}{rl}
 & x^2+6x+8 \\
\cline{2-2}
x-1) & x^3+5x^2+2x-8 \\
 & -x^3+x^2 \\
\cline{2-2}
 & 6x^2+2x \\
 & -6x^2+6x \\
\cline{2-2}
 & 8x-8 \\
 & -8x+8 \\
\cline{2-2}
 & 0
\end{array}
\tag{34}
$$

Since the remainder is zero, we know that $(+1)$ is a real root – we have also retrieved the quotient x^2+6x+8 which we can factor mentally $(x+2)(x+4)$. The remaining roots are thus (-2) and (-4).

2 A well-known logarithmic recurrence (not tested)

Many divide-and-conquer algorithms have a number of steps calculable with a recurrence that refers to a term $t(\frac{n}{2})$. This includes merge sort, quick sort and for you engineers, bisection method and Fast Fourier Transforms. This was mentioned in passing in class with the recurrence ...

$$T(0) = c \tag{35}$$

$$T(n) = T(\frac{n}{2}) + c \tag{36}$$

The solution to this recurrence is $c\lg n + c$. We will not go into its derivation in this class. If you are interested, I will provide you with a resource on this solution.

3 Combinatorics: An introduction and review to counting

The reasoning behind combinations and permutations is at the heart of counting. Let's reintroduce counting with a few exercises.

1. Books on a bookshelf

   (a) There are six books standing upright on a bookshelf. How many ways are there to arrange them on this bookshelf?

   (b) There is exactly one red book among your six books. How many arrangements of the six books are there such that the red book is furthest to the left?

   (c) There are exactly two blue books among your six books. How many arrangments of the six books are there such that the two blue books are standing together?

2. Numerals and letters

   (a) How many ten digit binary values have the character `1` in them?

(b) How many five digit hexadecimal values have the character `F` in them?

(c) How many six digit hexadecimal values have the substring `BEEF` in them?

We have to concern ourselves with these ideas in general when thinking about combinatorics ...

- If physical objects are being rearranged, then we exhaust the supply of such objects as we place them into spots (*books in a bookshelf*).
- If we are working with objects that are drawn limitlessly from a class of possible objects, then no such exhaustion takes place; we say that we are drawing the objects with replacement (*numerals and letters*).
- If the problem described can be broken into subproblems, the total count is often the product of the counts of each subproblem.

4 Lab Test 2 next week

The following material is on Lab Test 2 ...

- Strong mathematical induction
  - *Practice using the recurrence relationships discussed today.*
- Solving homogeneous linear recurrences
- Counting: Introduction to combinatorics

References

[1] Kenneth H. Rosen. *Discrete mathematics and its applications.* McGraw-Hill, 7th edition, 2012.