CIS2910 Discrete Structures II

Select Solutions for Labs 9 and 10! Oh my!

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1 Bayes' Theorem

1.1 Probability with Additional Information

There is a left box and a right box. In the left box there are 2 red blocks, 3 green blocks and 4 blue blocks. In the right box, there is 1 red block, 3 green blocks and 5 blue blocks. *Assume that you will draw from the left box half of the time and the right box the other half of the time*[1].

1. If we draw a green block from a randomly selected box, what is the probability it was drawn from the left box?

   Let's simplify this problem:

   $$p(L|G) = \frac{p(L \cap G)}{p(G)}$$

   where L is left, G is green, $\bar{L}$ is right, $\bar{G}$ is not green.

   given: $p(L \cap G) = p(G \cap L) = p(G|L)p(L)$ (commutativity, conditional probability)

   given: $p(G) = p(G \cap L) + p(G \cap \bar{L})$ (complement, distributivity)

   given: $p(G \cap \bar{L}) = p(G|\bar{L})p(\bar{L})$ (conditional probability)

   which results in Bayes' Theorem:

   $$p(L|G) = \frac{p(G|L)p(L)}{p(G|L)p(L) + p(G|\bar{L})p(\bar{L})}$$

   So we can just plug in our numbers and solve for each of these questions.

   $$p(L|G) = \frac{(\frac{3}{7})(\frac{1}{2})}{(\frac{3}{7})(\frac{1}{2}) + (\frac{3}{7})(\frac{1}{2})} = \frac{1}{2}$$

[1] This assumption was not clearly stated in the original exercise.

Note that you will not be asked to rederive Bayes' theorem, but you should know it well enough to apply it and answer a few questions about it. Please see §7.3 of Rosen 7th[1] if you want the complete derivation.

2. If we select a blue block, what is the probability it was drawn from the right box?

   Let B be the event of selecting a blue block.

$$p(\bar{L}|B) = \frac{(\frac{5}{7})(\frac{1}{2})}{(\frac{5}{7})(\frac{1}{2}) + (\frac{4}{7})(\frac{1}{2})} = \frac{5}{9}$$

3. If we select a red block, what is the probability it was drawn from the right box?

   Let R be the event of selecting a red block.

$$p(\bar{L}|R) = \frac{(\frac{1}{7})(\frac{1}{2})}{(\frac{1}{7})(\frac{1}{2}) + (\frac{2}{7})(\frac{1}{2})} = \frac{1}{3}$$

4. *Consider:* What would happen to the answers in questions $(1, 2, 3)$ if a person did not evenly draw from each of the two boxes equally as often? Say 40% of the time, this person would draw from the left box and 60% of the time from the right box.

5. *Consider:* What would happen to the answers in questions $(1, 2, 3)$ if there were an additional 10 white blocks but only in the right box?

1.2 Applying Bayes' Theorem

The City of Gruff installed a redlight camera at Collage and Garden two weeks ago. The team is fine tuning the device and have asked for your help. Each day, *1 in 20 drivers*[2] run a red light at that intersection.

1. The redlight is correct 80% of the time when it catches a car running the redlight. It is also correct 90% of the time when it identifies a car as proceeding lawfully through the intersection. What is the probability of a person identified as guilty actually having run the redlight? What is the probability that a person identified as law-abiding actually is actually law-abiding?

   This is a Bayes' Theorem question – let us state the variables and the probability values that we know – then we'll state the theorems for each case and solve for the unknown probabilities.

   Let R be the event that a car runs a red light, that $\bar{R}$ is the event that a does not run, that G is the event that a person is guilty, and that $\bar{G}$ that a person is not guilty.

$$\begin{aligned} p(R) &= 0.05 & p(\bar{R}) &= 0.95 \\ p(G|R) &= 0.8 & p(\bar{G}|R) &= 0.2 \\ p(G|\bar{R}) &= 0.1 & p(\bar{G}|\bar{R}) &= 0.9 \end{aligned}$$

[2]This value was incorrectly stated in the original exercise.

$$p(R|G) = \frac{p(G|R)p(R)}{p(G|R)p(R) + p(G|\bar{R})p(\bar{R})}$$

$$p(R|G) = \frac{(0.8)(0.05)}{(0.8)(0.05) + (0.1)(0.95)} = 0.2963$$

There is about a 30% probability that a person caught by the redlight camera is actually guilty of a crime.

$$p(\bar{R}|\bar{G}) = \frac{p(\bar{G}|\bar{R})p(\bar{R})}{p(\bar{G}|\bar{R})p(\bar{R}) + p(\bar{G}|R)p(R)}$$

$$p(\bar{R}|\bar{G}) = \frac{(0.9)(0.95)}{(0.9)(0.95) + (0.2)(0.05)} = 0.9884$$

There is about a 99% probability that a person deemed innocent by the redlight camera is actually innocent.

2. If the device is correct 50% of the time when identifying guilty parties, and 95% of the time when identifying law-abiding citizens, what are the probabilities then?

$$p(R|G) = \frac{(0.5)(0.05)}{(0.5)(0.05) + (0.05)(0.95)} = 0.3448$$

$$p(\bar{R}|\bar{G}) = \frac{(0.95)(0.95)}{(0.95)(0.95) + (0.5)(0.05)} = 0.9730$$

3. If the device is correct 99% of the time when identifying guilty parties, and 70% of the time when identifying law-abiding citizens, what are the probabilities then?

$$p(R|G) = \frac{(0.99)(0.05)}{(0.99)(0.05) + (0.3)(0.95)} = 0.1480$$

$$p(\bar{R}|\bar{G}) = \frac{(0.7)(0.95)}{(0.7)(0.95) + (0.01)(0.05)} = 0.9992$$

4. *Consider:* If the ratio of people that run a redlight is unknown, then we can estimate $p(R|G)$ by making $p(R) = p(\bar{R}) = 0.5$. This is referred to as a simplification of Bayes' theorem. How would this simplification affect the calculation of the above questions $(1, 2, 3)$?

   In the simplified Bayes' theorem, we would thus have:

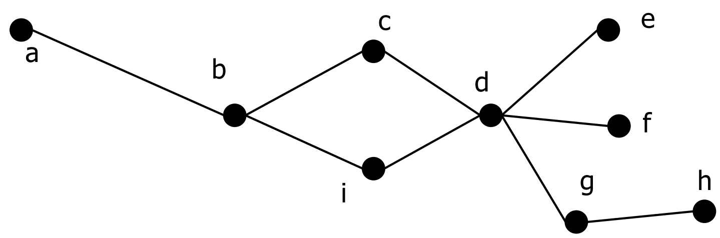
$$p(R|G) = \frac{p(G|R)p(R)}{p(G|R)p(R) + p(G|\bar{R})p(\bar{R})} \doteq \frac{p(G|R)(0.5)}{p(G|R)(0.5) + p(G|\bar{R})(0.5)} = \frac{p(G|R)}{p(G|R) + p(G|\bar{R})}$$

   Verify this result for yourself. Wouldn't this have been nice to know back in the section 1.1? Importantly, this simplification is only valid when we're missing $p(R)$.

2 Graph Theory

2.1 Introduction

1. Graph H.



   (a) Which vertex has the highest degree, and what degree has it?
   Vertex d with degree 5.

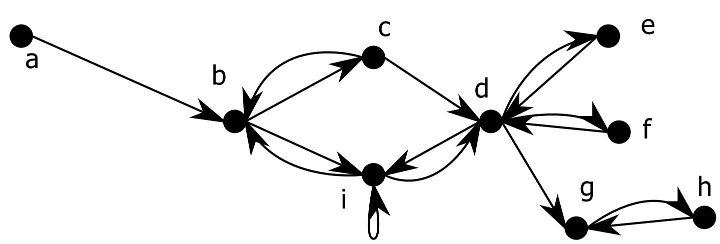
   (b) Which vertices are pendant?
   Vertices a, e, f, h.

   (c) There is a cycle in graph H. Write it down and indicate its size.
   Cycle b, c, d, i.

   (d) List all the neighbours of d.
   Vertices c, e, f, g, i

   (e) List all the neighbours of (b, c, d).
   Vertices a, e, f, g, i

2. Graph K.

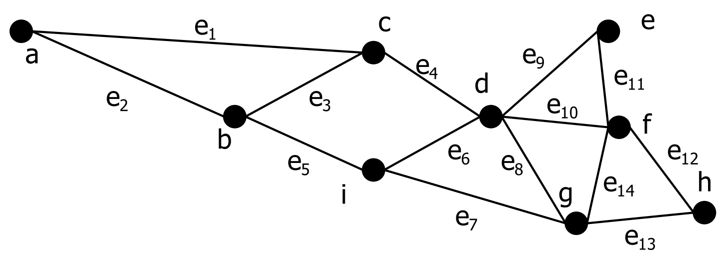


   (a) List the in-degrees and out-degrees of b, c and d in graph K.
   b, c, d have in-degrees $3, 1, 4$ respectively; and out-degrees $2, 2, 4$.

   (b) Which vertices are are adjacent from d in graph K?
   Vertices e, f, g, i.

   (c) Which vertices are adjacent to b in graph K?
   Vertices a, c, i.

3. Graph F.



(a) What is the adjacency list of the graph F?

initial vertex	terminal vertex
a	b, c
b	a, c, i
c	a, b, d
d	c, e, f, g, i
e	d, f
f	d, e, g, h
g	d, f, h, i
h	f, g
i	b, d, g

(b) What is the adjacency matrix of the graph F?

Let us index both rows and columns as $(a, b, c, d, e, f, g, h, i)$.

$$
\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} a & b & c & d & e & f & g & h & i \end{array} \\
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\end{array}
$$

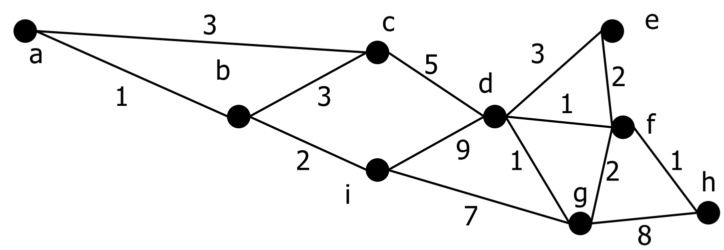
(c) What is the incidence matrix of the graph F?

Let us index the rows as $a, b, ..i$, and columns as $e_1, e_2, ..e_{14}$.

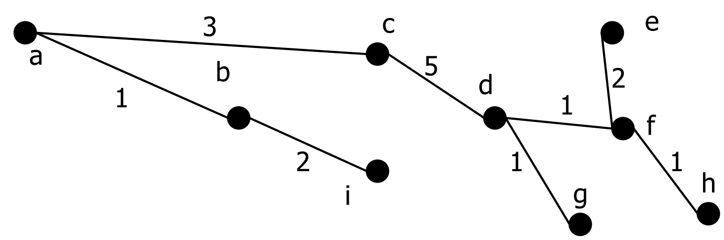
$$
\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{array}
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

2.2 Minimum Spanning Tree with Prim's Algorithm

Graph Z.



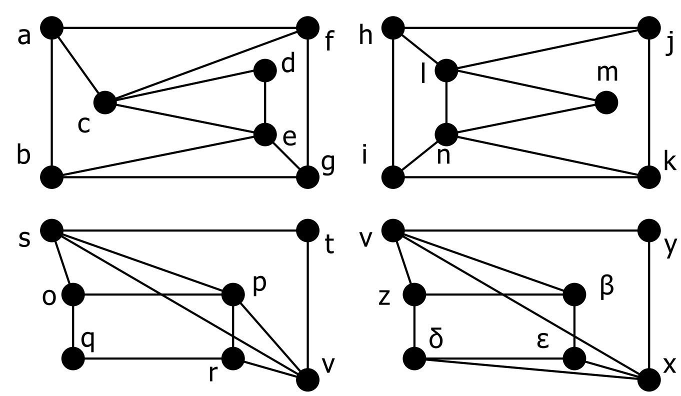
Draw a valid minimum spanning tree of graph Z (use Prim's Algorithm).



Note that when ties occur in the weight of the edge, I have chosen to break the tie with lexicographical (alphabetical) ordering.

2.3 Graph Problems

1. Graphs M, N, P, Q (clockwise starting from top left).



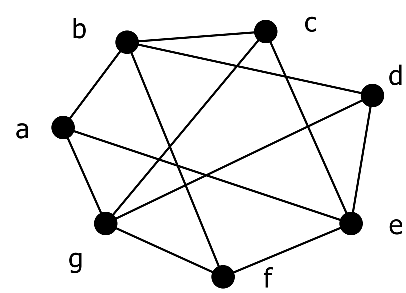
Are graphs M, N, P, Q isomorphic? What makes you say that these graphs are isomorphic or not isomorphic?

Graphs M, N, P are isomorphic but graph Q is not. The below isomorphism shows the mapping of vertices between M, N, P.

M	N	P
a	h	z
b	i	δ
c	l	v
d	m	y
e	n	x
f	j	β
g	k	ϵ

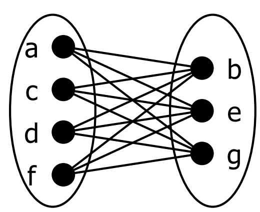
Note that Q is not isomorphic because no such mapping exists. We can quickly exclude Q since it has three vertices that have degree four, while M, N, P each have only two vertices that have degree four. Can you find any other differences in the degrees of Q as compared with M, N, P?

2. Graph S.



Is graph S bipartite? If so, redraw it to clearly show that this is true.

Yes. Here it is below.



References

[1] Kenneth H. Rosen. *Discrete mathematics and its applications*. McGraw-Hill, 7th edition, 2012.